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On the Asymptotic Behavior of Solutions of Certain Quasilinear Parabolic Equations

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0. INTRODUCTION

We are concerned here with the asymptotic behavior of solutions of certain types of reaction-diffusion equations. Consider, for example, the problem

$$\begin{aligned} u_t - \Delta u &= u^p, && x \in \Omega,\ t > 0 \\ u(x, 0) &= u_0(x), && x \in \Omega \\ u(x, t) &= 0, && x \in \partial\Omega,\ t > 0 \end{aligned} \tag{0.1}$$

with $\Omega \subseteq \mathbb{R}^N$ bounded, $p > 1$ and $u_0 \geqslant 0$. It is well known that there exist choices of u_0 for which the corresponding solutions tend to zero as $t \to \infty$, and other choices for which the solutions blow up in finite time. Indeed, the set of initial values for which one of these alternatives occurs is dense in $C_0^1(\bar{\Omega})$, say. (See [L].) We are interested in what other types of behavior may occur.

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Define the set of equilibrium solutions

$$E = \{w \in C^2(\Omega) \cap C_0(\bar{\Omega}) \colon w \geqslant 0 \text{ and } \Delta w + w^p = 0 \text{ in } \Omega\}.$$

Denote the solution of (0.1) by $u(x, t; u_0)$ and let

$$\omega(u_0) = \{w \in C_0(\bar{\Omega}) \colon \text{there exists a sequence } t_n \to \infty \text{ such that } u(\cdot, t_n; u_0) \to w \text{ uniformly}\}.$$

It is by now well established that if $u(x, t; u_0)$ is uniformly bounded for all $t \geqslant t_0 \geqslant 0$, then $\omega(u_0)$ is non-empty and in fact $\omega(u_0) \subseteq E$ [D, H, L]. For the problem at hand, E always contains $w \equiv 0$ and may or may not contain additional elements. However, any nonzero equilibrium state is dynamically unstable.

We may therefore consider that we have two stable states, $w \equiv 0$ and "$w \equiv \infty$." Based on topological considerations and analogies with finite-dimensional systems, one expects there to exist solutions of (0, 1) which exhibit some kind of "borderline" behavior, that is, they neither tend to zero nor blow up in finite time.

By the preceding remarks we may identify two further possibilities:

(I) $u(\cdot, t; u_0)$ converges to a nonzero (hence unstable) equilibrium solution of (0.1) along some subsequence.

(II) $u(\cdot, t; u_0)$ exists for all time but is not uniformly bounded.

Trivially, if $E \backslash \{0\}$ is nonempty, then (I) occurs with $u(\cdot, t; w) \equiv w$, where $w \in E \backslash \{0\}$. Results in [H] identify some circumstances under which this may also happen for initial values sufficiently close to a given equilibrium state on the so-called stable manifold. If $E \backslash \{0\}$ is empty, then just as clearly (I) is impossible. We know of no previous results concerning alternative (II).

Our results, as applied to this particular problem, may be expressed roughly in the following way.

Let Ω be a convex domain in $\mathbb{R}^N$.

(i) If $1 < p < (N + 2)/N$, then (II) cannot occur and (I) does occur.

(ii) If $p \geqslant (N + 2)/(N - 2)$, $N \geqslant 3$, then (I) cannot occur and (II) does occur.

Results related to ours are contained in the paper of Lions [L], one of the main results of which is that uniformly bounded solutions of equations of the type (0.1) tend to zero as $t \to \infty$, generically with respect to initial values. Thus he gives in some sense a picture of the infinite-dimensional flow induced by Eq. (0.1). Although not originally motivated by such considerations, our results give some further indication of what this picture looks like, in particular for those orbits which do not tend to zero as $t \to \infty$

(see Remark (iii) in Section 1). For a finite-dimensional analog, one might consider a two-dimensional autonomous system for which the first quadrant is positively invariant and 0 and ∞ are the only two stable rest states.

We do not restrict attention to classical solutions but instead allow for very weak solutions. This has the effect of increasing the generality of the results in most respects, although decreasing the generality in at least one respect; see Remark (v) in Section 1. The use of such solutions seems to arise in a natural way, given the nature of the estimates we prove. Of course, part of the interest is identifying circumstances when a weak solution must be a classical solution.

For a general discussion of semilinear parabolic equations in a functional setting, see the book of Henry [H] or Weissler [W]. Weissler has recently obtained some other results (in case $N=1$) related to those given here. The book of Henry also contains some analysis of the behavior of solutions near equilibrium solutions.

We actually consider equations which are not necessarily semilinear, namely, equations of the form

$$u_t = \Delta(\phi(u)) + f(u)$$

under certain conditions on ϕ and f. Since we will be dealing with L^1 solutions of such equations, the theory of nonlinear semigroups becomes relevant. The survey articles of Crandall [C] and Evans [E] explain the connection. Finally, we make frequent use of local estimates for solutions of linear parabolic equations of the type presented in the book by Ladyzenskaja *et al.* [LSU].

Section 1 contains a number of definitions, statements of the main theorems and more commentary. Section 2 contains a number of preliminary results, and the theorems are then proved in Section 3. In Section 4, we discuss some extensions of these results to degenerate parabolic equations.

1. Statements of Principal Results

Consider the problem

$$\begin{aligned} u_t &= \nabla \cdot (\sigma(u)\, \nabla u) + f(u), && x \in \Omega, t > 0 \\ u(x, 0) &= u_0(x), && x \in \Omega \\ u(x, t) &= 0, && x \in \partial\Omega, t > 0. \end{aligned} \tag{1.1}$$

Setting

$$\phi(u) = \int_0^u \sigma(s)\, ds,$$

the equation may also be written in the form

$$u_t = \Delta\phi(u) + f(u). \tag{1.2}$$

Set

$$\Omega_T = \Omega \times (0, T)$$
$$g(u) = f(\phi^{-1}(u))$$
$$G(u) = \int_0^u g(s)\, ds.$$

Denote by $\lambda_1 = \lambda_1(\Omega)$ the first eigenvalue of the Dirichlet problem

$$-\Delta\psi = \lambda\psi, \qquad x \in \Omega$$
$$\psi = 0, \qquad x \in \partial\Omega.$$

Denote the first eigenfunction by ψ_1; we may assume that $\psi_1 > 0$ in Ω and $\|\psi_1\|_{L^1(\Omega)} = 1$.

Let

$$V^2(\Omega_T) = L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)).$$

Several notions of solution of (1.1) will be employed in our discussion.

DEFINITIONS. (i) $u \in C([0, T]; L^1(\Omega))$ is an L^1 solution of (1.1) on $[0, T]$ if $f(u) \in L^1(\Omega_T)$ and

$$\int_s^t \int_\Omega [u\rho_t + \phi(u)\Delta\rho + f(u)\rho]\, dx\, dt - \int_\Omega [u\rho|_s^t]\, dx = 0$$

for every $\rho \in C^2(\bar{\Omega}_T)$, $\rho(x, t) = 0$ for $x \in \partial\Omega$ and $0 \leqslant s < t \leqslant T$.

(ii) If, in addition, $u \in V^2(\Omega_T)$, then we will call u a V^2 solution of (1.1) on $[0, T]$.

(iii) If also $u \in C^2(\Omega_T) \cap C^1(\bar{\Omega}_T)$, then we will call u a classical solution of (1.1) on $[0, T]$.

If $u_0 \in L^1(\Omega)$ and f is Lipshitz continuous, then global time L^1 solutions of (1.1) are obtainable via nonlinear semigroup theory [C, E]. If f is only locally Lipschitz and we take $u_0 \in L^\infty(\Omega)$, then a local time L^1 solution of (1.1) may be found by the same methods. If u is a given L^1 solution of (1.1), then u may be considered to be the semigroup solution of $u_t - \Delta\phi(u) = h$, where $h(x, t) = f(u(x, t)) \in L^1(\Omega_T)$ by assumption.

In the book [LSU], V^2 solutions as well as classical solutions are studied in great detail. In particular, a classical solution is always a V^2 solution, and if u is an L^1 solution and $u \in L^\infty(\Omega_T)$, then u is a classical solution on $[t_0, T]$ for any $t_0 > 0$ provided σ, f and $\partial\Omega$ are sufficiently smooth.

Here are some hypotheses we will use, not necessarily simultaneously.

(H_1) $\Omega \subseteq \mathbb{R}^N$ is convex, bounded with smooth boundary.

(H_2) $\sigma \in C^1(\mathbb{R}^+)$ and there exists $\delta > 0$ such that $\delta \leqslant \sigma(u) \leqslant \delta^{-1}$ for all $u \geqslant 0$.

(H_3) $f \in C^1(\mathbb{R}^+)$, $f(0) = 0$, f is convex and $0 \leqslant f'(0)/\sigma(0) < \lambda_1(\Omega)$.

(H_4) There exist constants $0 < A_1, A_2 < \infty$, $1 < p < (N+2)/N$ such that $f(u) \leqslant A_1 + A_2 u^p$ for all $u \geqslant 0$.

(H_5) g is convex on $\mathbb{R}^+$ with $\liminf_{s \to \infty} g(s)/s > \lambda_1(\Omega)$ and $\int_a^\infty (1/g(s))\, ds < \infty$ for some $a > 0$.

(H_6) $G(u) \leqslant [(N-2)/2N]\, ug(u)$ for all $u \geqslant 0$.

Now we state our main results.

THEOREM A. *Assume* (H_1)—(H_5) *hold.*

(i) *Any nonnegative global V^2 solution of* (1.1) *is uniformly bounded,* (*and hence classical*), *for all $t \geqslant t_0 > 0$.*

(ii) *If $\sigma(u) \equiv \sigma_0$ a constant, then any nonnegative global L^1 solution of* (1.1) *is uniformly bounded* (*and hence classical*), *for all $t \geqslant t_0 > 0$.*

(iii) *For any $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$ and $u_0 \not\equiv 0$, there exists $\tau > 0$ depending on u_0 so that $u(x, t; \tau u_0)$ is uniformly bounded on $[0, \infty)$, $\omega(\tau u_0) \neq \varnothing$ and $\omega(\tau u_0) \subseteq E \backslash \{0\}$.*

THEOREM B. *Assume* (H_1), (H_2) *hold.*

(i) *If* (H_6) *holds, $f \in C^1(\mathbb{R}^+)$ and u is a nonnegative L^1 solution of* (1.1) *which is uniformly bounded for $t \geqslant t_0 > 0$, then $u(\cdot, t) \to 0$ uniformly as $t \to \infty$.*

(ii) *If* (H_3), (H_5) *and* (H_6) *hold, then for any $0 \leqslant u_0 \in L^\infty(\Omega)$ with $u_0 \not\equiv 0$, there exists $\tau > 0$ depending on u_0 so that* (1.1) *has a global L^1 solution $u(x, t; \tau u_0)$ which is not uniformly bounded.*

Remarks. (i) In Theorem A(i) and (ii), we obtain a *uniform* L^∞ bound away from $t = 0$ for nonnegative global solution of (1.1). This is not quite an a priori bound, however, since it depends in a somewhat obscure way on the initial value u_0. This dependence does not involve any norm of u_0, but rather the "shape" of u_0 near $\partial\Omega$. Various authors have proved such a regularity result given some estimate for a lower norm of the solution and stronger assumptions on u_0 [LSU, R]. The point is that alternative (II) (in Section 0) can never happen under these circumstances. We also suspect that Theorem A(ii) holds for *all* nonnegative L^1 solutions of (1.1) without the assumption that $\sigma(u) \equiv$ constant.

(ii) Theorem A(iii) says that any ray in the positive cone of $L^\infty(\Omega)$ intersects an orbit leading into a nonzero equilibrium state. Under our hypotheses, such states are dynamically unstable (see, e.g., [L]), hence we have shown that the existence of a nontrivial stable "manifold." In the semilinear case, the existence and other properties of a local stable manifold near a given equilibrium state is studied in [H]. The result here gives some indication as to the "extent" of the stable manifold.

(iii) Combining Theorem A with some other recent results, we may obtain in some cases a fairly clear picture of the structure of the set of all nonnegative solutions of (1.1). To illustrate, let us take the special case of Eq. (0.1) with $1 < p < (N+2)/N$, and Ω a ball in $\mathbb{R}^N$. It is known that (0.1) has a unique positive equilibrium solution $w(x)$. (See [GNN].) Now, define

$$K = \{u_0 \in L^\infty(\Omega)\colon u_0 \geqslant 0 \text{ and } u(x, t; u_0) \text{ is uniformly bounded for all time}\}.$$

This set is the principal object of study in [L] where it is shown (for a more general class of equations) that if $u_0 \in K$ is not an extreme point of K, then $\omega(u_0) = \{0\}$. It follows that for $0 \not\equiv u_0 \geqslant 0$, there can exist at most one number $\tau > 0$ such that $\tau u_0 \in K$ and $0 \notin \omega(\tau u_0)$. Thus we have the following situation: for every $u_0 \in L^\infty(\Omega)$, $u_0 \geqslant 0$, $u_0 \not\equiv 0$, there exists a unique $\tau > 0$ such that $u(x, t; \tau u_0) \to w$ as $t \to \infty$. For $\gamma < \tau$, $u(x, t; \gamma u_0) \to 0$ exponentially (see [HD] for a more precise result) while for $\gamma > \tau$, $u(x, t; \gamma u_0)$ blows up in finite time in the L^1 sense. For the last assertion, we use a continuation result of Weissler [W, Corollary 3.2] which says that either $\lim_{t \to T^*} \|u(\cdot, t; \gamma u_0)\|_{L^1(\Omega)} = \infty$ for some $T^* < \infty$, or else $u(x, t; \gamma u_0)$ may be continued globally; Theorem A rules out the latter possibility.

(iv) The problem of the uniqueness of positive steady states for (1.1) is in general not fully understood at this time. For some very recent results, see Brézis and Nirenberg [BN], *Ni* [N] and *Ni* and Nussbaum [NN].

(v) Theorem B(i) follows simply from the fact that $\omega(u_0)$ consists of steady states and under these circumstances Pohozaev's result [P] implies that there are no positive steady states. Thus alternative (I) of Section 0 cannot happen.

We call attention at this point to same connections with results in a paper of Matano [M]. In the case $\sigma(u) \equiv$ constant, he has shown that (for very general nonlinearity f) if w_0 and w_1 are two equilibrium solutions of (1.1) with $w_0 \leqslant w_1$ and there is no other equilibrium state between w_0 and w_1, then it must be the case that the solutions of (1.1) with initial value u_0,

$w_0 \leqslant u_0 \leqslant w_1$, $u_0 \not\equiv w_0$, $u_0 \not\equiv w_1$, all tend either to w_0 or w_1 as $t \to \infty$. In particular, if there are two stable states w_0 and w_1 with $w_0 \leqslant w_1$, then there exists an unstable state w_2, $w_0 \leqslant w_2 \leqslant w_1$. Thus the region of attraction of w_0 and w_1 cannot together include all the functions between w_0 and w_1. A natural question, raised in [M], is what the corresponding situation is if the larger solution w_1 is replaced by ∞. In our setting we always have $w_0 \equiv 0$ stable in the usual sense, and we may regard "$w_1 \equiv \infty$" also as a stable state since, as we shall see, $\|u(\cdot, t)\|_{L^1(\Omega)} \to \infty$ if $\|u_0\|_{L^1(\Omega)}$ is sufficiently large. Now it is well known that there may or may not be positive equilibrium solutions, hence it is possible that the regions of attraction of 0 and ∞ do include all nonnegative initial values. Theorem B(ii) gives some indication as to the borderline behavior which occurs in the absence of positive equilibrium solutions, namely, there exist global time solutions of (1.1) which are not uniformly bounded. Thus alternative (II) of Section 0 does happen.

It would be interesting to know if such solutions are classical on every bounded time interval. The usual type of bootstrap argument cannot possibly work here, since if it did it would follow that the solution is actually uniformly bounded, while the proof will show it cannot tend to zero. This would contradict the first part of Theorem B. In any case, either u blows up in L^∞ sense in finite time but continues on in a weaker sense, or else u is classical for all time with

$$\lim_{t \to \infty} \| u(\cdot, t)\|_{L^\infty(\Omega)} = \infty;$$

either alternative is interesting.

(vi) We conclude this section by describing the main steps in the proofs of Theorems A and B for the special case of Eq. (0.1).

The first observation is that if u is a global solution of (0.1), then

$$\int_\Omega u(x, t)\, \psi_1(x)\, dx \leqslant \lambda_1^{1/(p-1)} \tag{1.3}$$

for all $t \geqslant 0$. The formal argument, which goes back to Kaplan [K], is as follows. Multiply (0.1) by $\psi_1(x)$ and integrate over Ω. Integration by parts yields

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega u(x, t)\, \psi_1(x)\, dx \\
&\quad = \int_\Omega u^p(x, t)\, \psi_1(x)\, dx - \lambda_1 \int_\Omega u(x, t)\, \psi_1(x)\, dx \\
&\quad \geqslant \left(\int_\Omega u(x, t)\, \psi_1(x)\, dx\right)^p - \lambda_1 \int_\Omega u(x, t)\, \psi_1(x)\, dx
\end{aligned}$$

by Jensen's inequality. If (1.3) fails for some t, then $\int_\Omega u(x,t)\,\psi_1(x)\,dx$ becomes infinite in finite time, a contradiction.

Next, from (1.3) we derive a uniform bound for $\|u(\cdot,t)\|_{L^1(\Omega)}$. To accomplish this, we use some slight modifications of results in [GNN] combined with the Hopf maximum principle to show that the solution "decreases near the boundary." It is here that the convexity of Ω is used.

Now regard u as a solution of the linear equation

$$u_t = \Delta u + a(x,t)\,u$$

with $a(x,t) = u^{p-1}(x,t)$. In the case that $1 < p < (N+2)/N$, we may use the regularity theory for linear parabolic equations [LSU] together with the L^1 estimate for u to derive a uniform L^∞ bound for u. This gives Theorem A(ii). We remark that analogous arguments have been used for studying solutions of elliptic equations in de Figueiredo *et al.* [DLN].

To continue, set

$$\begin{aligned} I(u_0) = \{\gamma > 0\colon & u(x,t;\gamma u_0) \text{ is uniformly bounded and} \\ & \lim_{t\to\infty} u(x,t;\gamma u_0) = 0\}. \end{aligned}$$

By the above remarks and some other simple arguments, one sees that for $u_0 \geqslant 0$, $u_0 \not\equiv 0$, $I(u_0)$ is open, nonempty and bounded above. If $\tau = \text{l.u.b.}\ I(u_0)$, then $u(x,t;\tau u_0) = \lim_{\gamma\uparrow\tau} u(x,t;\gamma u_0)$ is a global solution of (0.1) and $\tau \notin I(u_0)$. If $1 < p < (N+2)/N$, then $u(x,t;\gamma u_0)$ is bounded above independent of x, t and $\gamma < \tau$, hence $u(x,t;\tau u_0)$ is uniformly bounded, but does not tend to zero. Since any bounded solution must have ω-limit set contained in the set of equilibrium solutions of (0.1), we obtain Theorem A(iii).

Theorem B applies to (0.1) in the case $p \geqslant (N+2)/(N-2)$, $N \geqslant 3$. As remarked earlier, there are no positive classical equilibrium solutions by Pohozaev's identity [P] hence Theorem B(i) is immediate. The solution $u(x,t;\tau u_0)$ as defined above still exists in a suitable weak sense, and furthermore cannot be uniformly bounded, since otherwise we could show the existence of a nontrivial equilibrium state which would contradict Pohozaev's theorem. In this way Theorem B(ii) is proved.

2. Preliminary Results

We now prove or recall a series of propositions, each of which has some independent interest. These results contain most of the essential elements for the proofs of Theorems A and B.

PROPOSITION 1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain. Let σ, $f \in C^1(\mathbb{R})$, and $0 < \delta \leqslant \sigma(u) \leqslant \delta^{-1}$ for $u \in \mathbb{R}$. If u is a solution of* (1.1) *which is uniformly bounded for $t \geqslant t_0 > 0$, then $\omega(u_0)$ is a nonempty, compact connected invariant subset of $C_0(\bar{\Omega})$ and $\omega(u_0) \subseteq E$, where E denotes the set of equilibrium solutions of* (1.1).

Remark. We need not specify the sense in which u is a solution, since if it is uniformly bounded for $t \geqslant t_0$ it must be classical for $t > t_0$. The conditions on f and σ may be weakened in various ways; in particular, the proof we give here really only uses the fact that σ and f are locally Lipschitz continuous.

Proof. Without loss of generality we may assume $t_0 = 0$. Let u be a solution of (1.1) which is uniformly bounded. By the Nash–Moser theorem [LSU, p. 204], the functions $\{u(\cdot, t)\}_{t \geqslant 1}$ are uniformly bounded in $C_0^{\alpha}(\bar{\Omega})$ for some $\alpha > 0$ depending on N and δ. The conclusion that $\omega(u_0) \neq \varnothing$ follows immediately from the Arzelà–Ascoli theorem. The fact that $\omega(u_0)$ is compact, connected and invariant is standard; see [D, H, L].

It follows again from the Nash–Moser theorem that $\bar{\sigma}(x, t) = \sigma(u(x, t)) \in C_0^{\alpha, (\alpha/2)}(\overline{\Omega_T \backslash \Omega_1})$ for every $T > 1$ with the norm in this space being independent of T. The function $v = \phi(u)$ satisfies the equation

$$v_t = \bar{\sigma}(x, t)\, \Delta v + \bar{\sigma}(x, t)\, g(v). \tag{2.1}$$

The Schauder estimates for parabolic equations [F, LSU] imply that v belongs to $C^{2+\alpha, 1+(\alpha/2)}(\Omega_T \backslash \Omega_1)$ for every $T > 0$ with norm again independent of T.

Multiplying the equation for u by $\sigma(u)\, u_t$ and integrating over $\Omega \times (1, t)$, we conclude

$$\int_1^t \int_\Omega v_t^2 \, dx \, dt \leqslant C$$

where C is a constant independent of t. Since v_t is uniformly continuous, we must have $v_t(x, t) \to 0$ as $t \to \infty$ for all $x \in \Omega$ and hence he same is true for u_t.

It follows easily that if $u(\cdot, t_n) \to w$ as $t_n \to \infty$, then $w \in C_0(\bar{\Omega})$ and $\Delta \phi(w) + f(w) = 0$ in the sense of distributions and hence classically. Q.E.D.

PROPOSITION 2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain and assume* (H_2), (H_3) *hold. Define $A = \{u_0 \in L^\infty(\Omega)$: $u_0 \geqslant 0$, $u(x, t; u_0)$ is uniformly bounded on $\Omega \times (0, \infty)$ and $u(x, t; u_0) \to 0$ uniformly as $t \to \infty\}$. Then A is relatively open in $L^\infty(\Omega) \cap \{u_0 \geqslant 0\}$.*

As before, $u(x, t; u_0)$ denotes the solution of the problem (1.1). Under the present conditions $u(x, t; u_0)$ is defined unambiguously, i.e., a suitable uniqueness theorem holds [LSU].

Proof. We claim that there exists $\varepsilon_0 > 0$ such that if $u(x, t; u_0) \leqslant \varepsilon_0$ for some $t \geqslant 0$, then $u \to 0$ uniformly as $t \to \infty$.

Pick a domain $\hat{\Omega}$ such that $\Omega \subset\subset \hat{\Omega}$ and $f'(0)/\sigma(0) < \lambda_1(\hat{\Omega})$. Denote the first eigenfunction of $-\Delta$ by $\hat{\psi}_1$ with the normalization $\|\hat{\psi}_1\|_{L^\infty(\hat{\Omega})} = 1$. There then exists $\varepsilon_1 > 0$ such that for $\varepsilon < \varepsilon_1$ the function $v_\varepsilon = \phi^{-1}(\varepsilon\hat{\psi}_1)$ is a supersolution of (1.1). Setting

$$\varepsilon_2 = \operatorname*{Min}_{x \in \bar{\Omega}} v_{\varepsilon_1}(x),$$

it follows that if

$$\|u(\cdot, t_0; u_0)\|_{L^\infty(\Omega)} \leqslant \varepsilon_2$$

for some t_0, then

$$\|u(\cdot, t; u_0)\|_{L^\infty(\Omega)} \leqslant \phi^{-1}(\varepsilon_1)$$

for all $t \geqslant t_0$.

On the other hand, $v \equiv 0$ is an isolated solution of

$$\begin{aligned} -\Delta v = g(v) \qquad &\text{and} \quad v > 0 \text{ in } \Omega \\ v = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{2.2}$$

Specifically, any solution of (2.2) must satisfy

$$\int_\Omega \psi_1(g(v) - \lambda_1 v)\, dx = 0 \tag{2.3}$$

where $\lambda_1 = \lambda_1(\Omega)$ and ψ_1 is the corresponding first eigenfunction. Equation (2,3) is obtained by multiplying (2.2) by ψ_1 and the equation for ψ_1 by v and then subtracting. It now follows that if $v \geqslant 0$ and $\not\equiv 0$, then

$$\| g(v)/v \|_{L^\infty(\Omega)} > \lambda_1,$$

and this is impossible for $|v|_{L^\infty(\Omega)}$ sufficiently small, by hypothesis (H_3).

Thus if $u(x, t; u_0)$ is sufficiently small, it remains for all later times in a region in which no nonzero equilibrium states may be found, and hence $u(x, t; u_0) \to 0$ as $t \to \infty$ by Proposition 1. This proves the claim.

Now pick $u_0 \in A$; there exists T such that $\|u(\cdot, T; u_0)\|_{L^\infty(\Omega)} \leqslant \varepsilon_0/2$. Also there exists $\delta > 0$ depending on T, $\|u\|_{L^\infty(\Omega_T)}$ and the data such that if $\|u_0 - \hat{u}_0\|_{L^\infty} \leqslant \delta$ then $\|u - \hat{u}\|_{L^\infty(\Omega_T)} \leqslant \varepsilon_0/2$, where $\hat{u}$ denotes the solution of (1.1) with initial value $\hat{u}_0$. Allowing this for the moment, it follows immediately from the first half of the proof that $\hat{u} \to 0$ uniformly as $t \to \infty$ for all such $\hat{u}_0$, which completes the proof.

To prove the last claim we consider the problem satisfied by $z = u - \hat{u}$, namely,

$$\begin{aligned} z_t &= \Delta(\alpha(x, t)\, z) + \beta(x, t, z), && z \in \Omega, t > 0 \\ z(x, 0) &= u_0(x) - \hat{u}_0(x), && x \in \Omega \qquad (2.4) \\ z(x, t) &= 0, && x \in \partial\Omega, t > 0 \end{aligned}$$

where $\alpha(x, t) = [\phi(u(x, t)) - \phi(\hat{u}(x, t))]/|u(x, t) - \hat{u}(x, t)|$ and $\beta(x, t, z) = f(u(x, t)) - f(u(x, t) + z)$. Now α is positive and smooth as long as u, $\hat{u}$ remain bounded, and

$$|\beta(x, t, z)| \leqslant |z| \operatorname{Max}(f'(M), f'(M + z)) \equiv \gamma_M(z)$$

where $M = |u|_{L^\infty(\Omega_t)}$. The desired result now follows directly by comparison with the solution of the initial value problem $h' = \gamma_M(h)$, $h(0) = \delta$. Q.E.D.

PROPOSITION 3. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded, and assume* (H_2), (H_3) *and* (H_5). *Then there exists $C_0 < \infty$ depending only on Ω, σ and f such that if u is any nonnegative global L^1 solution of* (1.1) *then*

$$\int_\Omega u(x, t)\, \psi_1(x)\, dx \leqslant C_0$$

for all $t \geqslant 0$.

This is a special case of a result proved in [S, Sect. 5]. We should also refer the reader to the references cited there for earlier results of this type. We remark that it is only here that the convexity assumptions from (H_3) and (H_5) are used. By examining the proof of this proposition, or alternatively, by making use of comparison theorems, one sees that these conditions could be relaxed somewhat.

PROPOSITION 4. *Let w be a V^2 solution of the linear initial-boundary value problem*

$$\begin{aligned} w_t &= \nabla \cdot (a(x, t) \nabla w) + \beta(x, t)\, w + \gamma(x, t), && (x, t) \in \Omega_t \\ w(x, 0) &= w_0(x), \quad x \in \Omega && (2.5) \\ w(x, t) &= 0, \quad x \in \partial\Omega, t > 0 \end{aligned}$$

with $0<\delta\leqslant\alpha(x,t)\leqslant\delta^{-1}$ *for all* $(x,t)\in\Omega_T$, *and* $\beta,\ \gamma\in L^r(0,T;L^q(\Omega))$ *with* $1<r,q<\infty$ *satisfying*

$$\frac{1}{r}+\frac{N}{2q}<1.$$

Let $0\leqslant t_1<t_2<t_3\leqslant T$, *then*

$$\|w\|_{L^\infty(\Omega\times(t_2,t_3))}\leqslant C$$

where C *depends only on the quantities*

$$N,\Omega,\delta,t_2-t_1,t_3-t_1,q,r,$$
$$\|\beta\|_{L^r(t_1,t_3;L^q(\Omega))},\ \|\gamma\|_{L^r(t_1,t_3;L^q(\Omega))}\ \text{and}\ \|w\|_{L^1(\Omega\times(t_1,t_3))}. \tag{2.6}$$

Remark. This result expresses the fact that it is possible to obtain a local estimate of the L^∞ norm of a solution of (2.5) in terms of its L^1 norm and the other data; in particular, the estimate is independent of T and w_0. If the L^1 norm were replaced by L^2 norm, this would be a special case of Theorem 8.1 [LSU, p. 192]. The proof given there may be modified in an obvious way to obtain the corresponding estimate in terms of $\|w\|_{L^p}$ for any $p>1$. (See also Theorem 8.17 in [GT].) The fact that such an estimate is possible with $p=1$ (any $p>0$ in fact) is not a new result, but it does not seem to be written anywhere. The authors of [BCP] promise a fuller treatment of these matters. We therefore make no attempt to prove the sharpest result of this type.

This proposition may be proved in several ways. The argument given below was suggested to us by G. Lieberman. By making use of the previous results mentioned above it is sufficient to obtain an estimate for $\|w\|_{L^2(\Omega\times(t_2,t_3))}$ in terms of the quantities (2.6).

Proof. Pick $\zeta\in C^\infty(0,T)$ with $0\leqslant\zeta\leqslant1$ in $(0,T)$, $\zeta\equiv1$ on (t_2,t_3), $\zeta\equiv0$ on $[0,t_1]$. Multiply (2.5) by $w\zeta^{2s}$, $s>1$ to be chosen later. Performing standard manipulations, we obtain

$$\|w\zeta^s\|^2_{V^2((t_1,t_3)\times\Omega)}\leqslant C_1+C_2\int_{t_1}^{t_3}\int_\Omega w^2\zeta^{2s-1}\,dx\,dt \tag{2.7}$$

where C_1,C_2 depend only on (2.6). In the derivation of (2.7), we use the fact that $V^2((t_1,t_3)\times\Omega)$ is continuously embedded into $L^{2r'}(t_1,t_3;L^{2q'}(\Omega))$ for $1/2r'+N/4q'\geqslant N/4$. where $1/r+1/r'=1$, $1/q+1/q'=1$ [LSU, p. 74].

Set $k=(N+2)/N$, $k'=(N+2)/2$. The integral on the right may be considered to be

$$\int_A\int w^2\zeta^{2s-1+k'}\,d\mu \tag{2.8}$$

where $d\mu = \zeta^{-k'}\ dxdt$ and $A = \operatorname{supp} \zeta \cap \{t \leqslant t_3\}$. We use interpolation estimate (2.8) by

$$\| w\zeta^{(2s-1+k')/2} \|_{L^{2k}(d\mu)}^{2\theta} \| w\zeta^{(2s-1+k')/2} \|_{L^1(d\mu)}^{2(1-\theta)} \tag{2.9}$$

where $\frac{1}{2} = \theta/2k + (1-\theta)$. For $s > \frac{1}{2} + k'/2$, the second term in (2.9) is bounded in terms of $\| w\|_{L^1(\Omega \times (t_1, t_3))}$. Also, $(2s-1+k')\,k - k' = 2sk$. Thus the first term is

$$\| w\zeta^s \|_{L^{2k}(\Omega \times (t_1, t_3))}^{2\theta}.$$

From the embedding of V^2 mentioned above and the fact that $\theta < 1$, it follows that

$$\| w\zeta^s \|_{V^2(\Omega \times (t_1, t_3))}$$

is estimated in terms of the quantities (2.6), and hence the same is true for $\| w\|_{L^2(\Omega \times (t_2, t_3))}$.

Q.E.D.

PROPOSITION 5. *Let $\Omega \subseteq \mathbb{R}^N$ be bounded smooth, σ, $f \in C^1(\mathbb{R})$ and $|f(u)| \leqslant A_1 + A_2 |u|^p$ for some $p \in (1, (N+2)/N)$ and $A_1, A_2 < \infty$. Suppose u is a V^2 solution of (1.1) on $[0, T]$ or else $\sigma(u) \equiv \sigma_0$, a constant, and u is an L^1 solution of (1.1) on $[0, T]$. Let $0 \leqslant t_1 < t_2 < t_3 \leqslant T$. Then*

$$\| u \|_{L^\infty(\Omega \times (t_2, t_3))} \leqslant C$$

where constant C depends only on

$$N, p, A_1, A_2, \delta, t_2 - t_1, t_3 - t_1, \| u \|_{L^\infty(t_1, t_3; L^1(\Omega))}. \tag{2.10}$$

In particular, u is a classical solution on $[t_0, T]$ for any $t_0 > 0$.

Proof. First suppose that u is a V^2 solution of (1.1) on $[0, T]$. Set

$$\beta(x,t) = \begin{cases} \dfrac{f(u(x,t))}{u(x,t)} & \text{if } |u(x,t)| > 1 \\ 0 & \text{if } |u(x,t)| \leqslant 1 \end{cases}$$
$$\gamma(x,t) = \begin{cases} f(u(x,t)) & \text{if } |u(x,t)| \leqslant 1 \\ 0 & \text{if } |u(x,t)| > 1 \end{cases} \tag{2.11}$$

so that $f(u(x,t)) = \beta(x,t)\, u(x,t) + \gamma(x,t)$. Clearly $|\gamma(x,t)| \leqslant A_1 + A_2$ and $|\beta(x,t)| \leqslant A_1 + A_2 |u(x,t)|^{p-1}$ so that

$$\| \beta \|_{L^\infty(t_1, t_3; L^q(\Omega))} \leqslant A_1 + A_2 \ \| u \|_{L^\infty(t_1, t_3; L^1(\Omega))}$$

for some $q > N/2$. Thus $\beta, \gamma \in L^r(t_1, t_3; L^q(\Omega))$ for some $q, r \in (1, \infty)$ with $1/r + N/2q < 1$. (This is correct for $N \geqslant 2$; the case $N = 1$ must be treated separately.) The norms of β and γ are estimated in this space by the quantities (2.10), hence the conclusion follows directly from Proposition 4.

Now suppose that $\sigma(u)$ is a constant, say, $\sigma(u) \equiv 1$ (without loss of generality), and u is an L^1 solution of (1.1) on $[0, T]$. We will show that $u \in L^\infty(\Omega \times (t_0, T))$ for any $t_0 > 0$. It then follows, in particular, that u is a V^2 solution of (1.1) on $[t_0, T]$ for any $t_0 > 0$, and the desired conclusion follows from the first part of the Proposition.

We may first of all regard u as the solution of

$$\begin{aligned} u_t &= \Delta u + \gamma(x, t), && (x, t) \in \Omega_t \\ u(x, 0) &= u_0(x), && x \in \Omega \\ u(x, t) &= 0, && x \in \partial\Omega, t > 0 \end{aligned} \tag{2.12}$$

with $\gamma(x, t) = f(u(x, t)) \in L^1(\Omega_T)$ by assumption. It is easily seen by the method of solving the adjoint problem that (2.12) has at most one L^1 solution. hence u must coincide with the known solution

$$u(x, t) = \int_\Omega G(x, t; \xi, 0)\, u_0(\xi)\, d\xi + \int_0^t \int_\Omega G(x, t; \xi, \tau)\, \gamma(\xi, \tau)\, d\xi\, d\tau \tag{2.13}$$

where G is the Green's function for $Lu = u_t - \Delta u$ in Ω with zero boundary conditions. The first term in (2.13) belongs to $L^\infty(\Omega)$ for every $t > 0$. We also have the following estimate for $G([A\,])$

$$\|G(x, t; \cdot, \cdot)\|_{L^q(\Omega_T)} \leqslant C$$

for any $q \in [1, (N+2)/N)$, $(x, t) \in \Omega_T$ for some constant C depending on N, Ω and q. By the symmetry properties of G, we have

$$\|G(\cdot, \cdot; \xi, \tau)\|_{L^q(\Omega_T)} \leqslant C \tag{2.14}$$

for the same range of q and $(\xi, \tau) \in \Omega_T$. Denoting the second term in (2.13) by $\bar{u}(x, t)$, we have by Jensen's inequality

$$|\bar{u}(x, t)|^q \leqslant \|\gamma\|_{L^1(\Omega_T)}^{q-1} \int_0^t \int_\Omega |G(x, t; \xi, \tau)|^q\, |\gamma(\xi, \tau)|\, d\xi\, d\tau.$$

Integrating with respect to (x, t) over Ω_T and using (2.14) shows that $\bar{u} \in L^q(\Omega_T)$ for any such q. Therefore u belongs to $L^q(\Omega \times (\tau, T))$ for any $\tau > 0$ and $q < (N+2)/N$. Replacing $u(x, t)$ by $u(x, t + \tau)$ for some $\tau \in [0, t_0)$, we may assume that $u \in L^q(\Omega_T)$ and $u_0 \in L^q(\Omega)$ for any fixed $q < (N+2)/N$.

We next regard u as the solution of

$$\begin{aligned} u_t - \Delta u &= \beta u + \gamma, && (x, t) \in \Omega_T \\ u(x, 0) &= u_0(x), && x \in \Omega \\ u(x, t) &= 0, && x \in \partial\Omega, \ t > 0 \end{aligned} \tag{2.15}$$

with β, γ defined in (2.11). For $q \in (p, (N+2)/N)$ we have $\beta \in L^{q'}(\Omega_T)$, $q' = q/(q-1)$. Thus the problem (2.15) may be properly set in $X = C([0, T]; L^1(\Omega)) \cap L^q(\Omega_T))$ for such q. We show (i) (2.15) has at most one solution from X and (ii) if $u_0 \in L^q(\Omega)$, then (2.15) has a solution z in X which satisfies $z \in L^\infty((t_0, T) \times \Omega)$ for any $t_0 > 0$. This will complete the proof.

First the uniqueness assertion. Suppose u, $\hat{u}$ are two solutions from X of (2.15). The difference $v = u - \hat{u}$ satisfies

$$\int_0^T \int_\Omega v[\psi_t - \Delta\psi + \beta\psi] \, dx \, dt = 0 \tag{2.16}$$

for every $\psi \in C^2(\bar{\Omega}_T)$, $\psi(x, T) = 0$ for $x \in \Omega$ and $\psi(x, t) = 0$ for $x \in \partial\Omega$ and $t > 0$. Let $\beta_n \in C^\infty(\bar{\Omega}_T)$, $\beta_n \to \beta$ in $L^{q'}(\Omega_T)$ and let $\zeta \in C_0^\infty(\Omega_T)$. Define ψ_n to be the solution of

$$\begin{aligned} \psi_t + \Delta\psi + \beta_n \psi &= \zeta, && (x, t) \in \Omega_T \\ \psi(x, T) &= 0, && x \in \Omega \\ \psi(x, t) &= 0, && x \in \partial\Omega. \end{aligned}$$

Each ψ_n is smooth, and since $q' > (N+2)/2$, we have $\|\psi_n\|_{L^\infty(\Omega_T)}$ uniformly bounded [LSU, Theorem 7.1, p. 181]. We may therefore take $\psi = \psi_n$ in (2.16) yielding

$$\int_0^T \int_\Omega v\zeta \, dx \, dt = \int_0^T v(\beta_n - \beta) \, \psi_n \, dx \, dt.$$

But

$$\lim_{n \to \infty} \int_0^T \int_\Omega v(\beta_n - \beta) \, \psi_n \, dx \, dt = 0$$

since $\beta_n \to \beta$ in $L^{q'}(\Omega_T)$ and $v \in L^q(\Omega_T)$. Since ζ is arbitrary, $v \equiv 0$ in Ω_T.

Finally, we construct a solution of (2.15). Define β_n as before, let $\gamma_n \in C^\infty(\bar{\Omega}_T)$, $\gamma_n \to \gamma$ a.e. and

$$\|\gamma_n\|_{L^\infty(\Omega_T)} \leqslant \|\gamma\|_{L^\infty(\Omega_T)}.$$

and let $u_{on} \in C_0^\infty(\Omega)$ with $u_{on} \to u_o$ in $L^q(\Omega)$. Define z_n to be the classical solution of

$$\begin{aligned} z_t &= \Delta z + \beta_n z + \gamma_n, && (x,t) \in \Omega_T \\ z(x,0) &= u_{on}(x), && x \in \Omega \\ z(x,t) &= 0, && x \in \partial\Omega, t > 0. \end{aligned} \tag{2.17}$$

Multiplying the equation by z^{q-1} and doing standard calculations gives

$$\| z_n^{q/2} \|_{V^2(\Omega_t)} \leqslant C$$

where C depends on $\|u_0\|_{L^q(\Omega)}$, $\|\beta\|_{L^{q'}(\Omega)}$, $\|\gamma\|_{L^\infty(\Omega_T)}$, N and Ω.

In particular the sequence $\{z_n\}$ is uniformly bounded in $L^\infty(0, T; L^q(\Omega))$. From Proposition 4 it follows that $\|z_n\|_{L^\infty(\Omega \times (t_0, T))}$ is uniformly bounded for any $t_0 > 0$. By Theorem 10.1, of [LSU, p. 204] the sequence $\{z_n\}$ is actually equicontinuous on $[t_0, T] \times \bar{\Omega}$. It is therefore easy to check that there is a subsequence of $\{z_n\}$ converging to a function $z \in X$ which is a solution of (2.15). Q.E.D.

To state the last proposition, we need some ideas and definitions from [GNN].

Given a bounded smooth domain Ω and a direction $\gamma \in \mathbb{R}^N$, consider the hyperplanes T_λ given by $x \cdot \gamma = \lambda$. For large positive λ, T_λ is disjoint from Ω, and as λ is decreased eventually a value $\lambda_0 = \lambda_0(\gamma, \Omega)$ is reached such that $T_{\lambda_0} \cap \partial\Omega \neq \varnothing$. For $\lambda < \lambda_0$ and λ near λ_0, the hyperplane cuts off a piece of Ω which we denote $\Sigma(\lambda, \gamma)$. Define $\Sigma'(\lambda, \gamma)$ to be the reflection of $\Sigma(\lambda, \gamma)$ across T_λ. As λ is further decreased, we eventually reach a value $\lambda_1 = \lambda_1(\gamma, \Omega)$ such that either $\Sigma'(\lambda_1, \gamma)$ is internally tangent to $\partial\Omega$ or else T_{λ_1} intersects $\partial\Omega$ somewhere orthogonally. For $\lambda \in [\lambda_1, \lambda_0)$ we call $\Sigma(\lambda, \gamma)$ a *cap* corresponding to the direction γ; the cap $\Sigma(\lambda_1, \gamma)$ is called the *maximal cap* for the direction γ. For any $x \in \mathbb{R}^N$ we denote by x^λ the reflection of x across T_λ.

PROPOSITION 6. *Let Ω be bounded, smooth and u be a nonnegative classical solution of* (1.1) *on* $[0, T]$ *with $u_0(x) \in C^1(\bar{\Omega})$. Fix a direction γ and define the caps $\Sigma(\lambda, \gamma)$ as above, for $\lambda \in [\lambda_1, \lambda_0)$. Let $\lambda \in (\lambda_1, \lambda_0)$ and*

$$u_0(x) < u_0(x^\lambda) \quad \text{and} \quad \nabla u_0 \cdot \gamma < 0 \quad \text{for all } x \in \Sigma(\lambda, y), \tag{2.18}$$

then for all $x \in \Sigma(\lambda, \gamma)$ and $0 \leqslant t \leqslant T$, we have

$$u(x,t) < u(x^\lambda, t) \quad \text{and} \quad \nabla_x u(x,t) \cdot \gamma < 0. \tag{2.19}$$

Remark. This is stated in [GNN, Theorem 5.2]. It is actually assumed there that (2.18) holds for all $\lambda \in (\lambda_1, \lambda_0)$, but an examination of the proof shows that it need only hold for some λ; we then of course deduce a correspondingly weaker conclusion.

3. Proofs of the Principal Results

Proof of Theorem A(i), (ii). Under the given hypotheses it follows from Proposition 5 that u is classical on $[t_0, T] \times \Omega$ for any $0 < t_0 < T < \infty$. By the Hopf maximum principle [F], $(\partial u/\partial n)(x_0, t_0) < 0$ for $x_0 \in \partial\Omega$ where $\partial/\partial n$ denotes the outer normal derivative.

By Proposition 3, $t \to \int_\Omega u(x, t)\, \psi_1(x)\, dx$ is uniformly bounded on $\mathbb{R}^+$. We wish to show that actually

$$t \to \int_\Omega u(x, t)\, dx$$

is uniformly bounded for $t \geqslant t_0$.

For each $x_0 \in \partial\Omega$, let $n(x_0)$ be the corresponding outer normal direction. Since Ω is convex, there exist numbers $\lambda_0(x_0)$, $\lambda_1(x_0)$ such that for $\lambda \in (\lambda_1(x_0), \lambda_0(x_0))$ the cap $\Sigma(\lambda, n(x_0))$ is nonempty, $x_0 \in \partial\Sigma(\lambda, n(x_0))$ and $\Sigma(\lambda_1(x_0), n(x_0))$ is the maximal cap corresponding to the direction $n(x_0)$. There exists a neighborhood $S(x_0)$ of x_0 in Ω such that $\nabla u \cdot n(x_0) < 0$ for $x \in S(x_0)$ and $t = t_0$. We pick $\bar{\lambda}(x_0)$ sufficiently close to $\lambda_0(x_0)$ so that $\Sigma(\bar{\lambda}(x_0), n(x_0)) \cup \Sigma'(\bar{\lambda}(x_0), n(x_0)) \subseteq S(x_0)$.

Let (y', y_N) denote local coordinates centered at x_0 with y_N in the direction $n(x_0)$. We may pick $\varepsilon > 0$ so small that the cylinder $C(x_0) = \{y: |y'| < \varepsilon, |y_N| < \varepsilon\}$ has the property that the reflection of $\overline{C(x_0) \cap \Omega}$ across $T_{\bar{\lambda}(x_0)}$ is compact in Ω.

Now the collection of cylinders $\{C(x_0)\}_{x_0 \in \partial\Omega}$ make up an open cover of $\partial\Omega$, hence

$$\partial\Omega \subseteq C(x_1) \cup C(x_2) \cup \cdots \cup C(x_m)$$

for some finite collection of points $\{x_i\}$, $x_i \in \partial\Omega$ and $i = 1, \ldots, m$. Let

$$C^0 = \Omega \Big\backslash \left(\bigcup_{i=1}^{m} C(x_i) \right),$$

it is clear that $C^0 \subseteq \Omega$. Finally, let $C'(x_i)$ be the reflection of $C(x_i) \cap \Omega$ across the associated $T_{\bar{\lambda}(x_i)}$.

By construction, we have $u(x, t_0) < u(x^{\bar{\lambda}(x_i)}, t_0)$ if $x \in \Sigma(\bar{\lambda}(x_i), n(x_i))$.

Hence by Proposition 6, the same property continues to hold for all $t \geqslant t_0$, and in particular for $x \in C(x_i) \cap \Omega$. Therefore, for $t \geqslant t_0$,

$$\begin{aligned}\int_\Omega u(x,t)\,dx &\leqslant \sum_{i=1}^m \int_{C(x_i)\cap\Omega} u(x,t)\,dx + \int_{C^0} u(x,t)\,dx \\ &\leqslant \sum_{i=1}^m \int_{C'(x_i)} u(x,t)\,dx + \int_{C^0} u(x,t)\,dx \\ &\leqslant (m+1)\int_{\Omega_0} u(x,t)\,dx\end{aligned}$$

where Ω_0 is a relatively compact subdomain of Ω containing C^0 and each $C'(x_i)$, $i = 1, 2, \ldots, m$. Letting $a_0 = \inf_{x\in\Omega_0} \psi_1(x)$, $a_0 > 0$ and we obtain

$$\int_\Omega u(x,t)\,dx \leqslant \frac{m+1}{a_0}\int_\Omega u(x,t)\,\psi_1(x)\,dx \leqslant \frac{m+1}{a_0} C_0$$

where C_0 is the constant from Proposition 3. The desired conclusion now follows from Proposition 5. Q.E.D.

Proof of Theorem A(iii). Fix $u_0 \in L^\infty(\Omega)$ and consider the family of problems

$$\begin{aligned} u_t &= \nabla\cdot(\sigma(u)\nabla u) + f(u), && x\in\Omega, t>0 \\ u(x,0) &= \gamma u_0(x), && x\in\Omega \\ u(x,t) &= 0, && x\in\partial\Omega, t>0. \end{aligned} \tag{3.1}$$

For any $\gamma > 0$, (3.1) has at least a local time solution, classical for $t > 0$, which we denote by u_γ. Let

$$\tau = \sup\{\gamma\in\mathbb{R}^+ : u_\gamma \text{ exists for all } t\geqslant 0 \text{ and } u_\gamma \to 0 \text{ uniformly as } t\to\infty\}.$$

By Proposition 2, $\tau > 0$ and by Proposition 3, $\tau < \infty$. If we can show that u_τ is uniformly bounded on $[0,\infty)\times\Omega$, then it follows from Proposition 1 that $\omega(\tau u_0) \neq \varnothing$, but it also follows from the definition of τ and Proposition 2 that $0 \notin \omega(\tau u_0)$.

To show that u_τ is uniformly bounded, it suffices to show that there exists a constant C such that

$$\|u_\gamma\|_{L^\infty(\Omega\times(0,\infty))} \leqslant C$$

for all $\gamma < \tau$, since then it is easy to check that $u_\tau(x,t) = \lim_{\gamma\to\tau} u_\gamma(x,t)$.

First, by comparison with the O.D.E.

$$h' = f(h)$$
$$h(0) = \tau \|u_0\|_{L^\infty(\Omega)},$$

it is clear that there exists $t_0 > 0$ and $C_1 < \infty$ such that

$$\|u_\gamma\|_{L^\infty(\Omega_{t_0})} \leqslant C_1$$

for all $\gamma \leqslant \tau$. Thus by Proposition 5 it is sufficient to show that there exists $C_2 < \infty$ such that

$$\|u_\gamma(\cdot, t)\|_{L^1(\Omega)} \leqslant C_2$$

for $t \geqslant t_0/2$ and $\gamma < \tau$. We obtain such a bound from the proof of the first part of the theorem, provided the sets C^0, $C(x_i)$ may be chosen independently of $\gamma < \tau$.

To see that this is the case, we first observe that since the sequence u_γ is monotonically increasing for each fixed x and t, we need only consider γ sufficiently close to τ. But then standard arguments [LSU] show that $\|u(\cdot, t)\|_{C^1(\bar{\Omega})}$ depends continuously on $\|u(\cdot, 0)\|_{L^\infty(\Omega)}$ for each $t > 0$. It follows that the sets $S(x_0)$ may be chosen so that $\nabla u_\gamma \cdot n(x_0) < 0$ for $t = t_0$ and all $\gamma \leqslant \tau$, γ sufficiently close to τ.

The rest of the construction proceeds independently of the particular solution, and in this way we derive a bound for $\|u_\gamma(\cdot, t)\|_{L^1(\Omega)}$ which is independent of $t \geqslant t_0/2$ and $\gamma < \tau$. This completes the proof. Q.E.D.

Proof of Theorem B(i). This is an immediate consequence of Proposition 1 and the result of Pohozaev [P] which states that the elliptic problem

$$\Delta u + g(u) = 0 \qquad \text{in} \quad \Omega$$
$$u = 0 \qquad \text{on} \quad \partial\Omega$$

has no positive classical solution if Ω is starlike with respect to some point and $G(u) \leqslant ((N-2)/2N)\, ug(u)$. Q.E.D.

Proof of Theorem B(ii). Define the functions u_γ and the number $\tau \in (0, \infty)$ as in the proof of Theorem A(iii). As there, we have a uniform bound for $\|u_\gamma(\cdot, t)\|_{L^1(\Omega)}$ for all $t \geqslant 0$ and $\gamma < \tau$. Recall that each u_γ is classical for $t > 0$. Multiplying its equation by ψ_1 and integrating over $\Omega \times (t_1, t_2)$ gives

$$\int_{t_1}^{t_2} \int_\Omega f(u_\gamma)\, \psi_1 \, dx\, dt \leqslant \int_\Omega u_\gamma(x, t_2)\, \psi_1(x)\, dx + \lambda_1 \int_{t_1}^{t_2} \int_\Omega \phi(u_\gamma)\, \psi_1 \, dx\, dt.$$

Since ϕ is globally Lipschitz, it follows that the term

$$\int_{t_1}^{t_2}\int_\Omega f(u_\gamma)\,\psi_1\,dx\,dt$$

is bounded independently of $\gamma < \tau$ for any fixed t_1, t_2.

Since f is nondecreasing, we may then estimate

$$\int_{t_1}^{t_2}\int_\Omega f(u_\gamma)\,dx\,dt$$

in terms of

$$\int_{t_1}^{t_2}\int_\Omega f(u_\gamma)\,\psi_1\,dx\,dt,$$

as was done for u_γ itself in the proof of Theorem A(i) and (ii).

The sequence u_γ is increasing in γ for each fixed x, t, so we may define

$$u^*(x,t) = \lim_{\gamma\to\tau} u_\gamma(x,t).$$

By Fatou's Lemma and the Monotone Convergence Theorem, $u^*(\cdot, t) \in L^1(\Omega)$ for all $t \geqslant 0$ and $u_\gamma(\cdot, t) \to u^*(\cdot, t)$ strongly in $L^1(\Omega)$ for every $t > 0$. Likewise $\phi(u_\gamma) \to \phi(u^*)$, $f(u_\gamma) \to f(u^*)$ strongly in $L^1(\Omega \times (t_1, t_2))$ for any $0 \leqslant t_1 < t_2 < \infty$.

It is then easy to see that

$$\int_{t_1}^{t_2}\int_\Omega [u^*\rho_t + \phi(u^*)\,\Delta\rho + f(u^*)\,\rho]\,dx\,dt - \int_\Omega [u^*\rho|_{t_1}^{t_2}]\,dx = 0$$

for every $\rho \in C^2(\bar\Omega \times [0,T])$ such that $\rho(x,t) = 0$ for $x \in \partial\Omega$. We have already observed $f(u^*) \in L^1(\Omega_T)$ for any $t > 0$. If we verify that $u^* \in C([0,T]; L^1(\Omega))$ for any $T > 0$, then u^* is an L^1 solution of (1.1) with $u^*(x,0) = \tau u_0(x)$.

To see that $u^* \in C([0,T]; L^1(\Omega))$, let $\hat u$ denote the solution of

$$\begin{aligned} \hat u_t &= \Delta\phi(\hat u) + f(u^*), && x \in \Omega,\ t > 0 \\ \hat u(x,0) &= \tau u_0(x), && x \in \Omega \\ \hat u(x,t) &= 0, && x \in \partial\Omega,\ t > 0. \end{aligned}$$

Since $f(u^*) \in L^1(\Omega_T)$ for any $T > 0$, this problem has a global time solution in the sense of nonlinear semigroups [C, E]; in particular, $\hat u \in C([0,T]; L^1(\Omega))$ for every $T > 0$. We also have, for $\gamma < \tau$,

$$\|\hat u(\cdot,t) - u_\gamma(\cdot,t)\|_{L^1(\Omega)} \leqslant (\tau-\gamma)\|u_0\|_{L^1(\Omega)} + \int_0^t \|f(u^*) - f(u_\gamma)\|_{L^1(\Omega)}\,ds$$

and thus letting $\gamma \to \tau$, we have $\hat u \equiv u^*$.

Finally, if u^* is uniformly bounded, then u^* is the unique classical solution of (1.1) with initial value $\tau u_0(x)$ and must have nonempty ω-limit set by Proposition 1. Now, part (i) implies that $u(x, t; \tau u_0) \to 0$ as $t \to \infty$. By the definition of τ and Proposition 2, we obtain a contradiction. Q.E.D.

4. Extensions to Degenerate Parabolic Equations

We discuss briefly here the possibility of extending our results to the case of degenerate diffusion equations. For definiteness let us consider the problem

$$\begin{aligned} &u_t - \Delta u^m = \lambda u^p, && x \in \Omega, t > 0 \\ &u(x, t) = 0, && x \in \partial\Omega, t > 0 \\ &u(x, 0) = u_0(x) \geqslant 0, && x \in \Omega \end{aligned} \tag{4.1}$$

with $m > 1$, $p \geqslant 1$ and $\lambda > 0$. If $1 \leqslant p \leqslant m$, the asymptotic behavior of all solutions of (4.1) is described in [S2]. When $p > m$, the general picture resembles that of the case $m = 1$, $p > 1$; namely, if u_0 is sufficiently small, the solution decays to zero, while if u_0 is sufficiently large, the solution blows up in finite time (see [S2]). We will consider this case only.

It is well known that (4.1) does not admit classical solution in general. The notion of L^1 solution may be used without change, and by V^2 solution we shall mean that $u^m \in V^2(\Omega_T)$. If $u \in L^\infty(\Omega_T)$, then $u \in C(\Omega_T)$, $u^m \in V^2(\Omega_T)$ and u is classical in a neighborhood of any point (x_0, t_0) at which $u(x_0, t_0) > 0$. (See [S1, S2] and the references given there.)

In trying to adapt the arguments of Theorems A and B to this problem, the lack of uniform parabolicity in the equation causes difficulties in several places. For simplicity, let us therefore consider a more special situation. We take Ω to be a ball in $\mathbb{R}^N$ and restrict attention to solutions which are radially symmetric and decreasing, that is, $u = u(r, t)$ with $u_r(r, t) \leqslant 0$ for all r and t. For such solutions the conclusion of Proposition 6 is always valid.*

We now state a theorem which may be proved by roughly the same methods as those used in the proofs of Theorems A and B. Afterwards we will describe the main points of difference.

Set

$R(\Omega) = \{u_0 \in L^\infty(\Omega): 0 \leqslant u_0 \not\equiv 0,\ u_0$ is radially symmetric and decreasing$\}$.

Theorem C. *Let Ω be a ball in $\mathbb{R}^N$.*

* See notes added in proof.

(i) *Suppose* $1 < m < p < (N+2)/N$. *If* $u(x, t)$ *is a global* V^2 *solution of* (4.1) *with* $u_0 \in R(\Omega)$, *then* u *is uniformly bounded.*

(ii) *Let* $u_0 \in R(\Omega)$, $1 < m < p < (N+2)/N$. *Then there exists* $\tau > 0$, *depending on* u_0, *such that* $u(x, t; \tau u_0)$ *is uniformly bounded and* $\lim_{t \to \infty} u(x, t; \tau u_0) = w(x)$ *uniformly, where* w *is the unique positive equilibrium solution of* (4.1).

(iii) *Let* $p/m \geqslant (N+2)/(N-2)$, $N \geqslant 3$. *If* u *is a nonnegative solution of* (4.1) *which is uniformly bounded, then* $\lim_{t \to \infty} u(x, t) = 0$ *uniformly.*

(iv) *Let* $p/m \geqslant (N+2)/(N-2)$, $N \geqslant 3$. *If* $u_0 \in R(\Omega)$ *then there exists* $\tau > 0$ *depending on* u_0 *such that* (4.1) *has a global solution* $u(x, t; \tau u_0)$ *which is not uniformly bounded.*

The proof of Theorem C is in outline the same as that for the nondegenerate case. Here are the necessary facts to account for in modifying the various elements of the proofs.

To adapt Proposition 1 to the present situation, we need first of all the precompactness of bounded orbits in $C(\bar{\Omega})$. This follows from the principal result of [S1, Theorem 1.1]; see also [S2, Theorem 2.2] and references given there. Given this fact, an argument due to Langlais and Phillips [LP] may be used to show that $\omega(u_0) \subseteq E$ whenever $u(x, t; u_0)$ is uniformly bounded. The proof of Proposition 2 is easily seen to be valid in this case, taking into account the above regularity result. Proposition 3 is valid as stated for this equation [S1, Theorem 5.1]. It is easy to see that if $u_0 \in R(\Omega)$ then $u(\cdot, t) \in R(\Omega)$ for all $t > 0$ for which $u(\cdot, t)$ is defined; hence the conclusion of Proposition 6 will hold as remarked earlier.

Finally, we need a local estimate as in Proposition 4. Note first that if u is a V^2 solution of (4.1), then $\bar{u} = (u-1)^-$ is a nonnegative subsolution of the nondegerate equation

$$w_t = \nabla \cdot (\alpha(x, t) \nabla w) + \beta(x, t) w + \gamma(x, t) \tag{4.2}$$

with $\alpha(x, t) = \max(m, m u^{m-1})$, and

$$\beta(x, t) = \gamma(x, t) = \lambda u^{p-1}(x, t).$$

Any local estimate for solutions of (4.2) will also be valid for nonnegative subsolutions. Now $\alpha(x, t)$ is not a priori bounded above; however, if we examine the proof of Theorem 8.1, of [LSU, p. 192] we see that $\|\alpha\|_{L^\infty}$ does not enter into the estimate if we localize in time only.

Thus one sees that if $0 \leqslant t_1 < t_2 < t_3 \leqslant T$ and u is a V^2 solution of (4.1) on Ω_T, then $\|\bar{u}\|_{L^\infty(\Omega \times (t_2, t_3))}$ is bounded by a constant depending only on N, Ω, m, $t_2 - t_1$, $t_3 - t_1$, p, $\|\beta\|_{L^r(t_1, t_3; L^q(\Omega))}$, $\|\gamma\|_{L^r(t_1, t_3; L^q(\Omega))}$ and $\|\bar{u}\|_{L^2(\Omega \times (t_1, t_3))}$.

The norms of β, γ are estimated as before, and we may replace the L^2

norm of $\bar{u}$ by the L^1 norm by arguing as in the proof of Proposition 4. Since $0 \leqslant \bar{u} \leqslant u$, we obtain the desired local estimate.

The remaining changes for the proof of Theorem C are not difficult and we leave the details to the reader.

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Note added in proof. The special assumptions made in Section 4 have enabled us to avoid the necessity of extending Proposition 6 and the Hopf boundary point lemma to the degenerate parabolic case. Indeed there can be no such extension of the Hopf lemma, since as is well known, if u_0 has compact support in Ω, the support of $u(\cdot, t)$ remains compact in Ω for some time interval.

It is, however, possible to prove a version of Proposition 6 by the use of approximation arguments. Namely, if $u_0 \in C_0(\bar{\Omega})$ and u_0 is nonincreasing in the direction γ in some cap $\underline{\Sigma}(\lambda, \gamma)$, then the same is true for $u(\cdot, t)$, $t > 0$. To see this, let $u_\varepsilon(x, t)$ be the solution of

$$\begin{aligned} u_{\varepsilon,t} &= \Delta(\varphi_\varepsilon(u_\varepsilon)) + u_\varepsilon^p, & x &\in \Omega,\ t > 0 \\ u_\varepsilon &= 0, & x &\in \partial\Omega,\ t > 0 \\ u_\varepsilon(x, 0) &= u_{0\varepsilon}(x), & x &\in \Omega \end{aligned}$$

where

$$\varphi_\varepsilon(u) = u^m + \varepsilon u$$

and $u_{0\varepsilon} \in C_0^1(\bar{\Omega})$ is chosen so that $\nabla u_{0\varepsilon} \cdot \gamma < 0$ in $\underline{\Sigma}(\lambda, \gamma)$; this can be done, for example, by adding $\varepsilon\psi_1$ to u_0 and then smoothing it so that it is in $C^1(\bar{\Omega})$ and is still strictly decreasing near $\partial\Omega$. If $u(x, t)$ exists for all $t \geqslant 0$ then on any time interval $[0, T]$, u_ε exists for ε sufficiently small ([L, S, U]). Then from Proposition 6 it follows that

$$\nabla u_\varepsilon(\cdot, t) \cdot \gamma < 0 \qquad \text{in} \quad \underline{\Sigma}(\lambda, \gamma),$$

and, in particular, $u_\varepsilon(\cdot, t)$ is nonincreasing in the direction γ for all t for which u_ε is defined. By straightforward modifications of the results in [S_1], $u_\varepsilon \to u$ as $\varepsilon \to 0$ uniformly on compact subsets of $\bar{\Omega} \times (0, \infty)$, which gives the desired conclusion.

We conclude that in the degenerate parabolic case we obtain results analogous to those in Theorem C in convex domains, provided we restrict our attention to initial values which decrease near $\partial\Omega$ in a suitable way.

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